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A UNIFIED APPROACH TO FRACTIONAL CALCULUS PERTAINING TO ALEPH-FUNCTIONS

V. G. GUPTA[1] & NAWAL KISHOR JANGID[2]

[1]Department of Mathematics, University of Rajasthan, Jaipur, Rajasthan, India

[2]Department of Mathematics, Swami Keshvanand Institute of Technology, Management and Gramothan, Ramnagaria, Jaipur, Rajasthan, India

ABSTRACT

In this paper we study a pair of unified and extended fractional integral operators involving the multivariable H-Function, **Aleph** -Function and general class of polynomials. During the course of our study, we establish five theorems pertaining to Mellin transforms of these operators. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of (known and new) fractional integral operators involving simpler functions can be obtained. For the sake of illustration, some special cases of our main result have been recorded here.

KEYWORDS: Multivariable H – Function, **Aleph** – Function, General Class of Polynomials, Fractional Integral, Mellin Transform

INTRODUCTION

We recall here the following definitions required for the present study:

(a) The general class of polynomials introduced and studied by Srivastava [14] is defined as

$$S_V^U[x]=\sum_{\ell=0}^{[V/U]}\frac{(-V)_{U\ell}A_{V,\ell}}{\ell!}x^{\ell} \tag{1.1}$$

where U, V are arbitrary positive integers and the coefficient $A_{V,\ell}(V,\ell\geq 0)$ are arbitrary constants, real or complex.

(b) The H-function of several complex variables, introduced and studied by Srivastava and Panda [16] is defined and represented in the following form:

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,\lambda:(u',v');\ldots;[u^{(r)},v^{(r)}]}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix}[(a):(\theta';\ldots;\theta^{r}]:[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}] \\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^{r}}\int_{L_1}\cdots\int_{L_r}U_1(s_1)\cdots U_r(s_r)V(s_1,\ldots,s_r)\,z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{1.2}$$

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \varphi_j^{(i)} s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=v^{(i)}+1}^{v^{(i)}} \Gamma(b_j^{(i)} - \varphi_j^{(i)} s_i)}, \quad \forall\, i \in \{1,\ldots,r\} \tag{1.3}$$

$$V(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\lambda} \Gamma\left(1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right)}{\prod_{j=\lambda+1}^{A} \Gamma\left(a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i\right) \prod_{j=1}^{C} \Gamma\left(1 - c_j + \sum_{i=1}^{r} \psi_j^{(i)} s_i\right)} \tag{1.4}$$

and $\omega = \sqrt{-1}$

For the conditions of existence on the several parameters of the H-function of several complex variables, we refer to H. M. Srivastava et al. [15, p. 251-253, Eqns. (c.2) to (c.8)].

(c) Siidland et al. [13](see also Saxena and Pogany[11]) defined and represented the **Aleph** -function in terms of Mellin-Barnes type integral as follows:

$$\aleph^{m,n}_{p_{i'},q_{i'},\tau_{i'};r'}[z] = \aleph^{m,n}_{p_{i'},q_{i'},\tau_{i'};r'}\left[z \left| \begin{matrix} (e_{j'},E_{j'})_{1,n}, \left[\tau_{i'}(e_{j'i'},E_{j'i'})\right]_{n+1,p_{i'};r'} \\ (f_{j'},F_{j'})_{1,m}, \left[\tau_{i'}(f_{j'i'},F_{j'i'})\right]_{m+1,q_{i'};r'} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_{i'},q_{i'},\tau_{i'};r'}(\xi)\, z^{-\xi} d\xi, \tag{1.5}$$

where $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_{i'},q_{i'},\tau_{i'};r'}(\xi) = \frac{\prod_{j'=1}^{m} \Gamma(f_{j'} + F_{j'}\xi) \prod_{j'=1}^{n} \Gamma(1 - e_{j'} - E_{j'}\xi)}{\sum_{i'=1}^{r'} \left\{ \tau_{i'} \prod_{j'=m+1}^{q_{i'}} \Gamma(1 - f_{j'i'} - F_{j'i'}\xi) \prod_{j'=n+1}^{p_{i'}} \Gamma(e_{j'i'} + E_{j'i'}\xi) \right\}} \tag{1.6}$$

For the conditions on the several parameters of the Aleph-function, one can refer to [5].

(d) The Mellin transform of f(x) will be denoted by M[f(x)] or by F(s). If p and y are real, we write

$s = p^{-1} + iy$. If $p \geq 1$, $f(x) \in Lp\,(0, \infty)$, then for

$$p = 1,\ M[f(x)] = F(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{1.7}$$

and

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}\,ds, \tag{1.8}$$

under suitable conditions on the variables and the parameters.

For p > 1,

$$M[f(x)]=F(s)=\ell.\text{i.m.}\int_{1/x}^{x} f(x)\,x^{s-1}dx, \tag{1.9}$$

where ℓ.i.m. denotes the usual limit in the mean for L_p-spaces.

(e) The pair of new extended fractional integral operators studied here is defined as

$$Q_{\gamma_n}^{\alpha,\beta}[f(x)]=tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}(x^t-y^t)\times$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;[u^{(r)},v^{(r)}]}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}[(a):(\theta';\ldots;\theta^{r}]:[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}]\\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right.\right]$$

$$\times\prod_{j=1}^{k}\aleph_{p_{i'j},q_{i'j},\tau_{i'j};r'}^{m_j,n_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\left|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j},\left[\tau_{i'j}(e_{j'i'j},E_{j'i'j})\right]_{n_j+1,p_{i'},j;r'}\\ (f_{j'j},F_{j'j})_{1,m_j},\left[\tau_{i'j}(f_{j'i'j},F_{j'i'j})\right]_{m_j+1,q_{i'j};r'r'}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r}S_{V_i}^{U_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy \tag{1.10}$$

and

$$R_{\gamma_n}^{\rho,\beta}[f(x)]=tx^{\rho}\int_x^{\infty}y^{-\rho-t\beta-1}(y^t-x^t)^{\beta}\times$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;[u^{(r)},v^{(r)}]}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}[(a):(\theta';\ldots;\theta^{r}]:[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}]\\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right.\right]$$

$$\times \prod_{j=1}^{k} \aleph_{p_{i'j}, q_{i'j}, \tau_{i'j}; r'}^{m_j, n_j} \left[z_j \left(\frac{y^t}{x^t}\right)^{a'_j} \left(1-\frac{y^t}{x^t}\right)^{b'_j} \left| \begin{matrix} (e_{j'j}, E_{j'j})_{1,n_j}, \left[\tau_{i'j}(e_{j'i'j}, E_{j'i'j})\right]_{n_j+1, p_{i'}, j; r'} \\ (f_{j'j}, F_{j'j})_{1,m_j}, \left[\tau_{i'j}(f_{j'i'j}, F_{j'i'j})\right]_{m_j+1, q_{i'j}; r'r'} \end{matrix} \right. \right]$$

$$\times \prod_{i=1}^{r} S_{V_i}^{U_i}\left[z_i \left(\frac{x^t}{y^t}\right)^{g_i} \left(1-\frac{x^t}{y^t}\right)^{h_i} \right] \psi\left(\frac{x^t}{y^t}\right) f(y)\, dy, \tag{1.11}$$

where $\nu=\left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu=\left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$ and t, u_i and v_i are positive numbers.

The kernels $\psi\left(\frac{y^t}{x^t}\right)$ and $\psi\left(\frac{x^t}{y^t}\right)$ appearing in (1.10) and (1.11) respectively, are assumed to be continuous functions such that the integrals make sense for wide classes of functions f(x).

The conditions for the existence of these operators are as follows :

(I) $f(x) \in Lp\,(0, \infty)$, (II) $1 \le p, q < \infty$, $p^{-1} + q^{-1} = 1$,

(III) $\mathrm{Re}(\alpha + \tau_{i'} t a'_j \frac{f_{j'i'j}}{F_{j'i'j}} + t \sum_{i=1}^{n} u^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}) > -q^{-1}$,

(IV) $\mathrm{Re}(\beta + \tau_{i'} t b'_j \frac{f_{j'i'j}}{F_{j'i'j}} + t \sum_{i=1}^{n} v^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}) > -q^{-1}$,

(V) $\mathrm{Re}(\rho + \tau_{i'} t a'_j \frac{f_{j'i'j}}{F_{j'i'j}} + t \sum_{i=1}^{n} v^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}) > -p^{-1}$,

where $j''=1, 2, \ldots, u^{(n)}$; $i' = 1, 2, \ldots r'$.

Condition (I) ensures that both operators defined by (1.10) and (1.11) belong to Lp $(0, \infty)$.

These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [1], Kober [3], Love [4], Saigo et al. [6], Saxena and Kumbhat [8, 9, 10], Goyal et al. [2], Saxena and Kiryakova [7], etc.

2. MAIN THEOREMS

Theorem 2.1: If $f(x) \in L_p(0, \infty)$, $1 \le p \le 2$, [or $f(x) \in M_p(0, \infty)$, p>2]

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\alpha+\tau_{i'}ta'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} u^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}>-q^{-1}\right],$$

$$\mathrm{Re}\left[\beta+\tau_{i'}tb'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} v^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}>-q^{-1}\right],$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, then

$$M\{Q_{\gamma_n}^{\alpha,\beta}[f(x)]\}=M\{f(x)\}R_{\lambda_n}^{\alpha-s+1,\beta}[1]. \tag{2.1}$$

where M_p □(0, ∞) stands for the class of all functions f(x) of L_p□(0, ∞) with p>2, which are inverse Mellin-transforms of the functions of L_p (-∞, ∞).

Proof: On taking Mellin transform of (1.10), we have

$$M\{Q_{\lambda_n}^{\alpha,\beta}[f(x)]\}=\int_0^{\infty} x^{s-1}\Big\{ tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}(x^t-y^t)^{\beta}\times$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;[u^{(r)},v^{(r)}]}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}[(a):(\theta';\ldots;\theta^{r}]:[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}] \\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right.\right]$$

$$\times\prod_{j=1}^{k} \aleph_{p_{i'j},q_{i'j},\tau_{i'j};r'}^{m_j,n_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\left|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j},\left[\tau_{i'j}(e_{j'i'j},E_{j'i'j})\right]_{n_j+1,p_{i'},j;r'} \\ (f_{j'j},F_{j'j})_{1,m_j},\left[\tau_{i'j}(f_{j'i'j},F_{j'i'j})\right]_{m_j+1,q_{i'j};r'r'}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r} S_{V_i}^{U_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy\Big\}dx. \tag{2.2}$$

Now, after changing the order of integration, which is permissible under the conditions stated, the result (2.1) follows easily in view of (1.11).

Theorem 2.2: If f(x) ∈ L_p (0, ∞), 1 ≤ p ≤ 2, [or f(x) ∈ M_p (0, ∞), p>2]

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\beta+\tau_{i'}tb'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} v^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right]>-q^{-1},$$

$$\operatorname{Re}\left[\rho+\tau_{i'}ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1}$$

for j″=1, 2, …, $u^{(n)}$ and the integrals present are absolutely convergent, then

$$M\ \{R^{\alpha,\beta}_{\gamma_n}[f(x)]\}=M\{f(x)\}Q^{\rho+s-1,\beta}_{\gamma_n}[1]. \tag{2.3}$$

Proof: On taking Mellin transform of (1.11), we have

$$M\ \{R^{\rho,\beta}_{\gamma_n}[f(x)]\}=\int_0^\infty x^{s-1}\left\{tx^{\rho}\int_x^\infty y^{-\rho-t\beta-1}(y^t-x^t)^{\beta}\right.$$

$$H^{0,\lambda:(u',v');\ldots;[u^{(r)},v^{(r)}]}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{array}{c}z_1\\ \vdots\\ z_r\end{array}\middle|\begin{array}{l}[(a):(\theta';\ldots;\theta^r]:[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}]\\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{array}\right]$$

$$\times\prod_{j=1}^{k}\aleph^{m_j,n_j}_{p_{i'j},q_{i'j},\tau_{i'j};r'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\middle|\begin{array}{l}(e_{j'j},E_{j'j})_{1,n_j},\left[\tau_{i'j}(e_{j'i'j},E_{j'i'j})\right]_{n_j+1,p_{i'},j;r'}\\ (f_{j'j},F_{j'j})_{1,m_j},\left[\tau_{i'j}(f_{j'i'j},F_{j'i'j})\right]_{m_j+1,q_{i'j};r'r'}\end{array}\right]$$

$$\left.\times\prod_{i=1}^{r}S^{U_i}_{V_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right)\text{f(y) dy}\right\}dx \tag{2.4}$$

Now, after changing the order of integration, the result (2.3) can be easily obtained in view of (1.10)

Theorem 2.3: If f(x) ∈ L_p (0, ∞), $p^{-1}+q^{-1}=1$, v(x) ∈ L_p (0, ∞),

$$\operatorname{Re}\left[\alpha+\tau_{i'}ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1},$$

$$\operatorname{Re}\left[(\beta+\tau_{i'}tb'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1}$$

for j″=1, 2, …, $u^{(n)}$ and the integrals present are absolutely convergent, then

$$\int_0^\infty v(x)Q^{\alpha,\beta}_{\gamma_n}[f(x)]dx=\int_0^\infty f(x)R^{\alpha,\beta}_{\gamma_n}[v(x)]dx\,. \tag{2.5}$$

Proof: The result (2.4) can be easily established in view of (1.10) and (1.11).

3. INVERSION FORMULAE

Theorem 3.1: If f(x) ∈ L_p (0, ∞), 1 ≤ p ≤ 2, [or f(x) ∈ M_p (0, ∞), p>2],

$$p^{-1}+q^{-1}=1, \mathrm{Re}(\alpha+\tau_{i'}ta'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}})>-q^{-1}, \mathrm{Re}\left[\beta+\tau_{i'}tb'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right]>-q^{-1},$$

for j″=1, 2, …, $u^{(n)}$ and the integrals present are absolutely convergent and

$$Q_{\gamma_n}^{\alpha,\beta}[f(x)=[v(x)], \tag{3.1}$$

then

$$f(x)=\int_0^\infty y^{-1}[v(y)]\left[h\left(\frac{x}{y}\right)\right]dy, \tag{3.2}$$

where

$$h(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}y^{-1}\frac{x^{-s}}{R(s)}ds, \tag{3.3}$$

and

$$R(x)=R_{\gamma_n}^{\alpha-s+1,\beta}\ [1]. \tag{3.4}$$

Proof: On taking Mellin transform of (3.1) and then applying Theorem 2.1, we get

$$M\{f(x)\}=\frac{M\{v(x)\}}{R(s)}$$

which on inverting leads to

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\frac{M\{v(x)\}}{R(s)}ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{R(s)}\left\{\int_0^\infty y^{s-1}[v(y)]dy\right\}ds.$$

Further, on changing the order of integration, we have

$$f(x)=\int_0^\infty y^{1}[v(y)]\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^{-s}\frac{1}{R(s)}ds\right\}dy.$$

Now, in view of (3.3), we easily arrive at (3.2).

Theorem 3.2: If $f(x) \in L_p(0, \infty)$, $1 \le p \le 2$, [or $f(x) \in M_p(0, \infty)$, $p>2$],

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\beta+\tau_{i'}tb'_{j}\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1},$$

$$\mathrm{Re}\left[\rho+\tau_{i'}ta'_{j}\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-p^{-1},$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, and

$$R^{\rho,\beta}_{\gamma_n}[f(x)=[w(x)], \tag{3.5}$$

then

$$f(x)=\int_0^\infty y^{-1}[w(y)]\left[G\left(\frac{x}{y}\right)\right]dy, \tag{3.6}$$

where

$$G(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{T(s)}ds, \tag{3.7}$$

and

$$T(x)=Q^{\rho+s-1,\beta}_{\gamma_n}\ [1]. \tag{3.8}$$

Proof: On taking Mellin transform of (3.5) and then applying Theorem 2.2, we get

$$M\{f(x)\}=\frac{M\{w(x)\}}{T(s)}$$

which on inverting leads to

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\frac{M\{w(x)\}}{T(s)}ds=\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{T(s)}\left\{\int_0^\infty y^{s-1}[w(y)]dy\right\}ds$$

Further, on changing the order of integration, we have

$$f(x)=\int_0^\infty y^{-1}[w(y)]\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^{-s}\frac{1}{T(s)}ds\right\}dy.$$

Now, (3.6) follows directly in view of (3.5).

4. GENERAL PROPERTIES

The properties given below are immediate consequences of the definitions (1.10) and (1.11)

$$x^{-1} Q_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = R_{\gamma_n}^{\alpha,\beta}[f(x)], \tag{4.1}$$

$$x^{-1} R_{\gamma_n}^{\rho,\beta}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = Q_{\gamma_n}^{\rho,\beta}[f(x)], \tag{4.2}$$

$$x^{u} Q_{\gamma_n}^{\alpha,\beta}[f(x)] = Q_{\gamma_n}^{\alpha-u,\beta}[x^{u} f(x)], \tag{4.3}$$

$$x^{u} R_{\gamma_n}^{\rho,\beta}[f(x)] = R_{\gamma_n}^{\rho+u,\beta}[x^{u} f(x)]. \tag{4.4}$$

The properties given below express the homogeneity of the operators Q and R respectively.

If $I_{\gamma_n}^{\alpha,\beta}[f(x)] = [\nu(x)]$ then

$$Q_{\gamma_n}^{\alpha,\beta}[f(cx)] = [\nu(cx)] \tag{4.5}$$

If $R_{\gamma_n}^{\rho,\beta}[f(x)] = [Z(x)]$, then

$$R_{\gamma_n}^{\rho,\beta}[f(cx)] = [Z(cx)]. \tag{4.6}$$

5. SPECIAL CASES

(i) Taking $\tau_{i'} = 1, i' = 1, \ldots, r'$ in (2.2) and (2.3) we get the results due to Srivastava et al [17].

(ii) Taking $\lambda = A = C = 0$, the multivariable H-function reduces to product of r Fox's H-functions and consequently we obtain the following results from (2.2) and (2.4)

$$M \ \{Q_{\lambda_n}^{\alpha,\beta}[f(x)]\} = \int_0^\infty x^{s-1} \left\{ tx^{-\alpha-t\beta-1} \int_0^x y^{\alpha} (x^t - y^t)^{\beta}\right.$$

$$\times \prod_{j=1}^{k} \aleph^{m_j, n_j}_{p_{i'j}, q_{i'j}, \tau_{i'j}; r'} \left[z_j \left(\frac{y^t}{x^t}\right)^{a'_j} \left(1-\frac{y^t}{x^t}\right)^{b'_j} \middle| \begin{matrix} (e_{j'j}, E_{j'j})_{1,n_j}, \left[\tau_{i'j}(e_{j'i'j}, E_{j'i'j})\right]_{n_j+1, p_{i'}, j; r'} \\ (f_{j'j}, F_{j'j})_{1,m_j}, \left[\tau_{i'j}(f_{j'i'j}, F_{j'i'j})\right]_{m_j+1, q_{i'j}; r'r'} \end{matrix} \right]$$

$$\prod_{i=1}^{r} H^{u^{(i)}, v^{(i)}}_{B^{(i)}, D^{(i)}} \left[z_i \middle| \begin{matrix} [b^{(i)}:\varphi^{(i)}] \\ [d^{(i)}:\delta^{(i)}] \end{matrix} \right] \times \prod_{i=1}^{r} S^{U_i}_{V_i} \left[z_i \left(\frac{y^t}{x^t}\right)^{g_i} \left(1-\frac{y^t}{x^t}\right)^{h_i} \right] \psi\left(\frac{y^t}{x^t}\right) f(y)\, dy \Bigg\} dx. \tag{5.1}$$

and $M\ \{R^{\rho,\beta}_{\gamma_n}[f(x)]\} = \int_0^\infty x^{s-1} \Big\{ t x^{\rho} \int_x^\infty y^{-\rho-t\beta-1} (y^t - x^t)^{\beta} \prod_{i=1}^{r} H^{u^{(i)}, v^{(i)}}_{B^{(i)}, D^{(i)}} \left[z_i \middle| \begin{matrix} [b^{(i)}:\varphi^{(i)}] \\ [d^{(i)}:\delta^{(i)}] \end{matrix} \right]$

$$\times \prod_{j=1}^{k} \aleph^{m_j, n_j}_{p_{i'j}, q_{i'j}, \tau_{i'j}; r'} \left[z_j \left(\frac{y^t}{x^t}\right)^{a'_j} \left(1-\frac{y^t}{x^t}\right)^{b'_j} \middle| \begin{matrix} (e_{j'j}, E_{j'j})_{1,n_j}, \left[\tau_{i'j}(e_{j'i'j}, E_{j'i'j})\right]_{n_j+1, p_{i'}, j; r'} \\ (f_{j'j}, F_{j'j})_{1,m_j}, \left[\tau_{i'j}(f_{j'i'j}, F_{j'i'j})\right]_{m_j+1, q_{i'j}; r'r'} \end{matrix} \right]$$

$$\times \prod_{i=1}^{r} S^{U_i}_{V_i} \left[z_i \left(\frac{x^t}{y^t}\right)^{g_i} \left(1-\frac{x^t}{y^t}\right)^{h_i} \right] \psi\left(\frac{x^t}{y^t}\right) f(y)\, dy \Bigg\} dx \tag{5.2}$$

6. CONCLUSIONS

The functions involved in the results established in this paper are unified and general in nature, hence a large number of known results lying in the literature follow as special cases. Further, on suitable specifications of the parameters involved, numerous new results involving simpler functions may also be obtained.

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REFERENCES

1. A. Erdélyi (1950-51). On some functional transformations, Univ. Politec. Torino, Rend. Sem. Mat., 10, 217-234.
2. S. P. Goyal, R. M. Jain and N. Gaur (1991). Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials, Indian J. Pure Appl. Math., 22[5], 403- 411. Acad. Phy. Sci., Vol. 10, No. 4. Pp. 373-382.
3. H. Kober (1940). On fractional integrals and derivatives, Quart. J. Math., 1, 193-211.

4. E. R. Love (1967). Some integral equations involving hypergeometric functions, J. Edinburgh Math. Soc., 3:15, 169-198.

5. J. Ram and D. Kumar (2011), Generalized fractional integration of the Aleph-function, J. Rajasthan

6. M. Saigo, R.K. Saxena and J. Ram (1992). On the fractional calculus operator associated with the H- function, Ganita Sandesh, 6(1), 36-47.

7. R. K. Saxena and V. S. Kiryakova (1992). On relation between the two-dimensional H- transforms in terms of Erdélyi -Kober operators, Math. Balkanica, 6, 133-140.

8. R. K. Saxena and R. K. Kumbhat (1974). Integral operators involving H-function, Indian J. of Pure Appl. Math., 5, 1-6.

9. R. K. Saxena and R. K. Kumbhat (1975). Some properties of generalized Kober operators, Vijnana Parishad Anusandhan Patrika, 18, 139-150.

10. R. K. Saxena and R. K. Kumbhat (1973). Fractional integration operators of two variables, Proc. Indian Acad. Sci., 78, 177-186.

11. R. K. Saxena and T. K. Pogany (2010), Mathieu-type series for the Aleph function occurring in Fokker-Planck equation, Eur. J. Pure Appl. Math. 3(6), 980-988.

12. V.P. Saxena (2008). The I-Function, Anamaya Publishers, New Delhi.

13. Siidland, N., Baumann, B. and Nonnenmacher, T.F. (1998). Open problem: who knows about the Aleph-function?, Fract. Calc. Appl. Anal. 1(4), 401-402.

14. H. M. Srivastava (1972). A contour integral involving Fox's H-function, Indian J. Math., 14, 1-6.

15. H. M. Srivastava, K.C. Gupta and S. P. Goyal (1982). The H-functions of one and two variables with applications, South Asian Publishers, New Delhi.

16. H.M. Srivastava, and R. Panda (1976). Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine, Angew, Math. 283/284, 265- 274.

17. A. Srivastava, R. Mukherjee and A. Bhargava (2014). A unified approach to fractional calculus pertaining to I-functions, TJPRC Pvt. Ltd. Vol. 4-1, 31-42